Nonlinear wave phenomena in Josephson junctions

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Table of contents

Literature

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Weak superconductivity. Josephson effect.

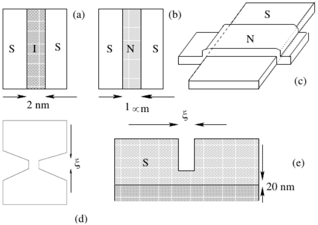
In 1962 Brian Josephson) predicted theoretically (experimental confirmation took place in 1963-64) the following facts:

- **DC Josephson effect**. Superconducting current can flow through the tunnel junction (for example superconductor/insulator/superconductor).
- **DC Josephson effect**. If the current exceeds certain critical value, the non-zero voltage drop appears on the junction which in its turn causes HF electromagnetic radiation.

In 1973 for his discovery Josephson was awarded the Nobel Prise.

Josephson equations

Weak superconductivity - a number of phenomena in systems that consist of weakly coupled superconductors separated by the media where supercondunductivity is absent or strongly supressed. Examples: (a) tunnel junction superconductor-insulator-superconductor (SIS); (b) superconductor-normal metal-superconductor (SNS) ; (c) Notarys bridge; (d-e) Dayem bridge.



- DC Josephson effect $I_s = I_c \sin\phi$, where $\phi = \Theta_2 - \Theta_1$ and $\Theta_{1,2}$ are the phases of the macroscopic wave functions of the superconductors 1 and 2; I_c - *critical* Josephson current that depends on the junction properties.
- AC Josephson effect

$$2eV = \hbar \frac{\partial \phi}{\partial t} \ . \tag{1}$$

Here V is the voltage drop on the junction.

Derivation of the Josephson equations.

Time evolution of the Josephson junction as a quantum mechanical system is governed by the Schrödinger equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi \qquad (2)$$

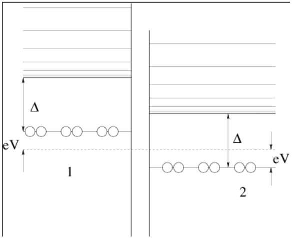
The wave-function is given as a linear combination of two states:
$\Psi(t) = \sum_{k=1,2} C_k(t)\psi_k$
Each superconductor is described by the macroscopic wave function $\psi_{1,2}$

$$i\hbar\frac{dC_k}{dt} = \sum_n H_{mn}C_n(t)\ ,\ H_{mn} = \int \psi_m^* \hat{H}\psi_n dV \qquad (3)$$

where $H_{11} = eV$, $H_{22} = -eV$, $H_{12} = H_{21} = K$.

$$i\hbar\frac{dC_1}{dt} = eVC_1(t) + KC_2(t), \qquad (4)$$

$$i\hbar\frac{dC_2}{dt} = KC_1(t) - eVC_2(t). \qquad (5)$$



Both superconducting leads are made from the same material: $|C_1|^2 = |C_2|^2 = n_s$, n_s - density of the supercurrent carriers, thus, $C_{1,2} = \sqrt{n_s}\exp(i\theta_{1,2})$. Separating the real and imaginary parts, we obtain:

$$\frac{dn_s}{dt} = \frac{2Kn_s}{\hbar}\sin\phi\,, \tag{6}$$

$$\frac{d\theta_1}{dt} = -\frac{K}{\hbar}\cos\phi - \frac{eV}{\hbar}\,, \tag{7}$$

$$\frac{d\theta_2}{dt} = -\frac{K}{\hbar}\cos\phi + \frac{eV}{\hbar}\,. \tag{8}$$

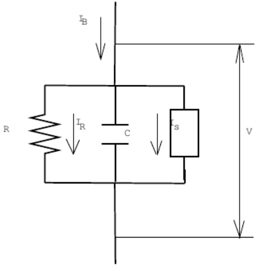
Here $\phi = \theta_2 - \theta_1$.

1. Subtracting Eqs. (7) and (8) we obtain $V = \frac{\hbar}{2e}\frac{d\phi}{dt}$.
2. Since the supercurrent through the junction satisfies $I_s \sim dn_s/dt$, we arrive to:

$$I_s = I_c \sin\phi\,. \tag{9}$$

Equivalent scheme of the small Josephson junction, RCSJ model

Within the RCSJ (resistively and capacitively shunted junction) model the DC biased JJ can be treated as a parallelly shunted (i) resistor, (ii) condensator and (iii) superconducting element. Use Kirchhoff laws:



$$I_B = I_Q + I_R + I_S,\ I_Q = \frac{dQ}{dt} = C\frac{dV}{dt}\ ,\ I_R = \frac{V}{R}.$$

$$\frac{C\hbar}{2e}\ddot{\phi} + \frac{\hbar}{2eR}\dot{\phi} + I_c \sin\phi = I_B\ ,$$

In the dimensionless form

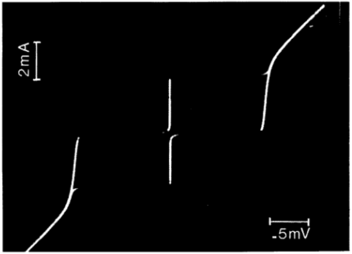
$$\ddot{\phi} + \alpha\dot{\phi} + \sin\phi = \gamma\ ,$$

$$t \to t\omega_J\ ,\omega_J = \sqrt{\frac{2eI_c}{C\hbar}}\ ,\alpha = \sqrt{\frac{\hbar}{2eI_cR^2C}}\ ,\ \gamma = I_B/I_c\ .$$

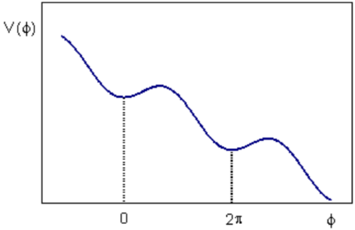
Current-voltage characteristic of the junction

- One of the ways to study the junction is to measure the current-voltage characteristic (CVC).
- Mechanical analogue - a pendulum with the constant torque γ or a particle in the washboard-like potential:

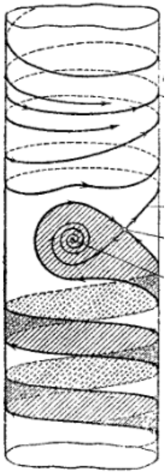
$$\ddot{\phi} + \alpha\dot{\phi} = -\frac{\partial V(\phi)}{\partial \phi}\;,\; V(\phi) = 1 - \cos\phi - \gamma\phi$$



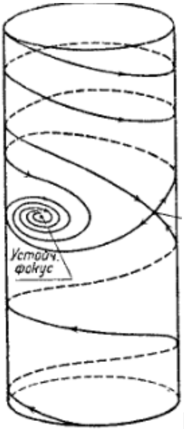
Experimental CVC, source: Barone, Paterno (1982).



CVC of the junction. Phase plane.



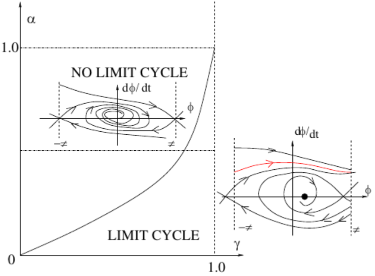
A limit cycle exists.



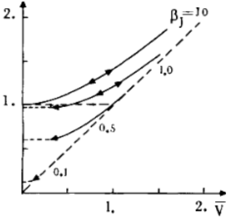
No limit cycle.

CVC of the junction. Hysteresis.

Depending on relation btw α and γ, on the phase plane $(\phi, \dot{\phi})$ there could exist a limit cycle
$\phi(t) = \phi_0 + \omega t + \psi(t)$,
$\psi(t) = \psi(t + T)$.



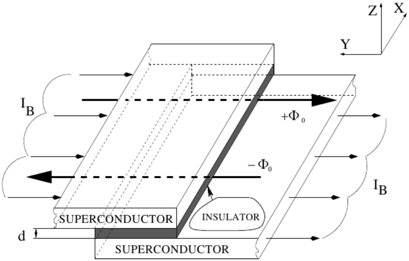
- For $\gamma_{thr} < \gamma < 1$ system can be in two states:
  - pendulum rotation with the constant frequency, $\bar{V} \propto \left\langle \dot{\phi} \right\rangle_T \neq 0$ - voltage drop on the junction -resistive state;
  - fixed point $\phi = arcsin\, \gamma$ - superconducting state.
- For $\gamma > 1$ - only resistive state.
- For $\gamma < \gamma_{thr}$ - only superconducting state.



by W.J. Johnson, PhD Thesis, Univ. of Wisconsin (1968); from Barone, Paterno (1982).

Long Josephson junctions

- Spatial dependence $\phi = \phi(x,t)$;
- Length in X direction $\gg$ length in Y direction, $l \gg w$;
- Thickness of the superconduncting leads $\gg$ London penetration depths ($\lambda_{1,2}$);
- Neglect boundary effects;
- Neglect dissipation effects and external currents.



$$\mathrm{rot}\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j} \Longrightarrow \frac{\partial H_y}{\partial x} = \frac{\partial D_z}{\partial t} + j_z = j_c \sin\phi + C/w \frac{\partial V}{\partial t} = j_c \sin\phi + \frac{C\hbar}{2ew}\frac{\partial^2 \phi}{\partial t^2} \,,$$

here j_c - critical current per unit area, $C = \epsilon\epsilon_0 w/d$ - capacitance per unit length.

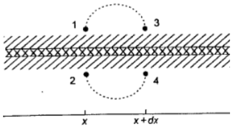
$$H_y = \pm \frac{\hbar}{2e(\lambda_1+\lambda_2+d)\mu_0}\frac{\partial\phi}{\partial x} \Longrightarrow \frac{1}{\omega_J^2}\phi_{tt} - \lambda_J^2\phi_{xx} + \sin\phi = 0 \,.$$

Sine-Gordon equation.

Long Josephson junction

Generalized momentum of the Cooper pair : $\hbar\nabla\theta = 2m\mathbf{v_s} + 2e\mathbf{A}$.

Ignore the areas where the supercurrent is absent ($\mathbf{v_s} \equiv 0$). Suppose $d + \lambda_1 + \lambda_2 \ll$ superconductor thickness



$$\int_1^3 \nabla\theta d\mathbf{l} + \int_2^4 \nabla\theta d\mathbf{l} = \frac{2e}{\hbar}\left(\int_1^3 \mathbf{A}d\mathbf{l} + \int_2^4 \mathbf{A}d\mathbf{l}\right) ,$$

- $$\frac{2e}{\hbar}\oint \mathbf{A}d\mathbf{l} \equiv \frac{2e}{\hbar}d\Phi = \theta_3 - \theta_1 + \theta_2 - \theta_4 ,$$
- Taking into account that $\theta_3 - \theta_4 = \phi(x + dx)$ and $\theta_1 - \theta_2 = \phi(x)$, one obtains

$$\frac{d\phi}{dx} = \frac{2e}{\hbar}\frac{d\Phi}{dx} = \frac{2\pi}{\Phi_0}\frac{d\Phi}{dx} , \quad \Phi_0 = \frac{\pi\hbar}{e}.$$

- Connection between the phase difference and magnetic field:

$$d\Phi = B_y(\lambda_1 + \lambda_2 + d)dx = \mu_0 H_y(\lambda_1 + \lambda_2 + d)dx \Longrightarrow$$
$$\Longrightarrow H_y(x,t) = \frac{\hbar}{2e(\lambda_1 + \lambda_2 + d)\mu_0}\frac{\partial\phi(x,t)}{\partial x} .$$

Electromagnetic excitations. Parameters.

- Josephson penetration depth - spatial scale of magnetic field penetration in X direction

$$\lambda_J = \sqrt{\frac{w\Phi_0}{2\pi\mu_0 I_c(d+\lambda_1+\lambda_2)}} = \sqrt{\frac{\hbar}{2eLI_c}} = \sqrt{\frac{\Phi_0}{2\pi L I_c}},\; L = \mu_0 \frac{d+\lambda_1+\lambda_2}{w}$$

- $\lambda_{1,2} \sim 10^{-7} m$, $\lambda_J \gg \lambda_{1,2}$: $\lambda_J \sim 0.1 mm$.
- Magnetic flux quantum:
  in SI: $\Phi_0 = \frac{\pi\hbar}{e} = 2.064^{-15} Weber$,
  in Gauss system : $\Phi_0 = \frac{\pi\hbar c}{e} = 2.064^{-7}\text{Gauss} \times \text{sm}^2$;
- Swihart velocity $\bar{c} = 1/\sqrt{LC} = c\sqrt{\frac{d}{\epsilon(d+\lambda_1+\lambda_2)}}$,
  maximal velocity of the electromagnetic waves velocity in the junction.
  For the typical parameters $\epsilon \simeq 4$, $d = 2\times 10^{-9}$m one obtains $\bar{c} \approx 0.02c \div 0.05c$.
- Josephson plasma frequency - minimal plane wave frequency:

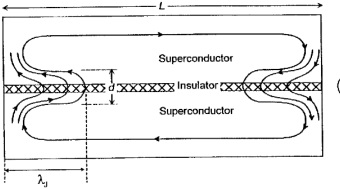
$$\omega_J = \bar{c}/\lambda_J = \sqrt{\frac{2\pi I_c}{C\Phi_0}}.$$

Electromagnetic excitations. Waves.

- Weak external field, $\phi \ll 1$

$$\omega_J^2 \phi_{tt} - \lambda_J^2 \phi_{xx} + \phi = 0 \ .$$

- Stationary case: $\phi \sim e^{-x/\lambda_J}$- magnetic field penetrates the junction in the depth $\sim \lambda_J$.



- Non-stationary case - small-amplitude waves (so-called Josephson plasmons) $\phi(x,t), H(x,t) \sim \exp i(qx - \omega t)$ with dispersion law:

$$\omega(q) = \pm\sqrt{\omega_J^2 + \bar{c}^2 q^2}.$$

- Nonlinear case - cnoidal waves, just nonlinear extension of plane waves, do not produce average voltage, $< V >_T = 0$.

Electromagnetic excitations. Vortices.

Travelling wave solutions

$$z = x - vt \ , \ \ u(x,t) = u(x - vt) \equiv u(z) \ .$$

$$\frac{d^2 u}{dz^2} = \frac{\sin u}{1 - v^2} \ .$$

SGE reduces to

$$\frac{v^2 - 1}{2} \left(\frac{du(z)}{dz} \right)^2 + (1 - \cos u) = E \ ,$$

$$z - z_0 = \pm \int_{u_0}^{u} \sqrt{\frac{v^2 - 1}{2\left[E - 2\sin^2(w/2)\right]}} dw \ .$$

Solutions can be expressed via elliptic functions

$$F(\phi; k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \equiv \zeta \ , \ \ k^2 < 1 \ , \phi = \mathrm{am}\zeta.$$

$$\mathrm{sn}(\zeta; k) = \sin \phi \ , \ \ \mathrm{cn}(\zeta; k) = \cos \phi \ ,$$

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^1 \frac{dx}{\sqrt{1 - x^2}\sqrt{1 - k^2 x^2}} \ .$$

Electromagnetic excitations. Vortices.

- When $H > H_{cr}$ - spiral waves:

$$\phi(x,t) = \pi + 2\arcsin\left[\pm \mathrm{sn}\left(\pm\frac{x - vt - x_0}{k\lambda_J\sqrt{1-(v/\bar{c})^2}}; k\right]\right)$$

$$V = \mp\frac{\Phi_0\omega_J}{2\pi}\frac{2v/\bar{c}}{k\sqrt{1-(v/\bar{c})^2}}\mathrm{dn}\left[\frac{x - vt - x_0}{\lambda_J\sqrt{1-(v/\bar{c})^2}}; k\right],$$

- Spiral waves are trains of vortices or, alternatively, sequences of positive (negative) pulses with the period $2k\sqrt{1-(v/\bar{c})^2}K(k)/\omega_J$, $\bar{V} \neq 0$!.
- For $k \to 1$ the period $\to \infty$ and just on vortex remains

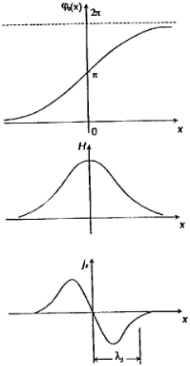
$$\phi(x,t) = 4\arctan\left[\exp\left(\pm\frac{x - vt}{\lambda_J\sqrt{1-(v/\bar{c})^2}}\right)\right].$$

- Vortex carries exactly one magnetic flux quantum:

$$\Phi = \int_{-\infty}^{+\infty} d\Phi = \frac{2\pi}{\Phi_0}\int_{-\infty}^{+\infty} d\phi = \pm\frac{\hbar}{2e}[\phi(+\infty) - \phi(-\infty)] = \pm\Phi_0 .$$

Therefore Josephson vortex is also called *fluxon*, and spiral waves - fluxon waves.

Electromagnetic excitations. Vortices.

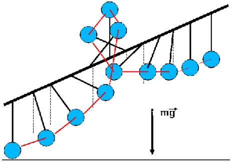


- Fluxon (+) or antifluxon(-) form a two-parametric (x_0, v) family of solutions

$$\phi(x,t) = 4\arctan\left[\exp\left(\pm\frac{x+x_0-vt}{\lambda_J\sqrt{1-(v/\bar{c})^2}}\right)\right].$$

- Velocity is bounded $|v| < \bar{c}$.
- Thickness is defined by the Lorentz contraction factor $\propto \sqrt{1-(v/\bar{c})^2}$.

Mechanical analogue



Electromagnetic excitations. Total energy.

Total energy stored in the finite junction.

$$w_J = \int_0^t P(t')dt' = \int_0^t I(t')V(t')dt' = \int_0^t \left[I_c \sin\phi + \frac{C}{d_1}\frac{\partial V}{\partial t'} \right] V dt' =$$
$$= \frac{I_c \hbar}{2e}\int_0^t \sin\phi \frac{\partial \phi}{\partial t'} dt' + \frac{C}{w}\left(\frac{\hbar}{2e}\right)^2 \int_0^t \phi(t')\frac{\partial \phi}{\partial t'}dt' = \frac{C}{w}\left(\frac{\hbar}{2e}\right)^2 \frac{\phi_t^2}{2} +$$
$$+ \frac{j_c \hbar}{2e}(1-\cos\phi)\ .$$

Magnetic field energy density:

$$w_H = \frac{\mu_0 H^2 d_z}{2} = \frac{\mu_0 d_z}{2}\left(\frac{\hbar}{2ed_z\mu_0}\right)^2 \phi_x^2\ , \quad d_z = d + \lambda_1 + \lambda_2\ .$$

Total free energy density $w = w_J + w_H$:

$$w = \frac{\hbar^2}{4\mu_0 e^2 d_z}\frac{\phi_x^2}{2} + \frac{C\hbar^2}{4e^2 w}\frac{\phi_t^2}{2} + \frac{j_c\hbar}{2e}(1-\cos\phi) = \frac{j_c\hbar}{2e}\left[\frac{\lambda_J^2\phi_x^2}{2} + \frac{\omega_J^2\phi_t^2}{2} + 1 - \cos\phi\right].$$

In dimensionless units:

$$W = \frac{j_c\hbar\lambda_J}{2e}\int_0^{L/\lambda_J}\left(\frac{\phi_x^2+\phi_t^2}{2} + 1 - \cos\phi\right)dx = \frac{j_c\hbar\lambda_J}{2e}\bar{W}\ .$$

Solitons and their properties

- Solitons - nonlinear wave excitations, that
  - are localized in space;
  - propagate with constant shape and velocity;
  - do not change when interact.
- Solitons are observed in many physical systems: surface waves, optical waveguides, magnets, cold atomic gases (Bose-Einstein condensates).
- Nonlinear wave equation that support solitons can be solved exacly through the Inverse Scattering Transform (IST).
- Sine-Gordon equation ($\phi_{tt} - \phi_{xx} + \sin\phi = 0$) is completely integrable - has infinite number of integrals of motion. Some of them:
  - Charge $Q = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi_x dx$,
  - Momentum $P = -\int_{-\infty}^{+\infty} \phi_t \phi_x dx$.
  - Energy $H = \int_{-\infty}^{+\infty} \left(\frac{\phi_t^2}{2} + \frac{\phi_x^2}{2} + 1 - \cos\phi\right) dx$.

Inverse scattering transform (IST) method

- "Direct" spectral problem

$$\psi_x = \hat{U}(\lambda)\psi \;, \quad \hat{U}(\lambda) = \begin{bmatrix} i\lambda & q(x) \\ r(x) & -i\lambda \end{bmatrix} \;, \quad \psi(x,\lambda) = (\psi_1, \psi_2)^T ,$$

  λ - spectral parameter, $q(x), r(x) \in C^\infty$, $\int_{-\infty}^{+\infty} |r(x)|dx < \infty$, $\int_{-\infty}^{+\infty} |q(x)|dx < \infty$, $q(x) = r(x) \equiv \phi_x/2$ for SG.
- For the spectral problem $\phi_x(x)$ is potential, for SGE - initial condition.
- Time evolution of scattering data

$$\psi_t = \hat{V}(\lambda)\psi \;, \quad \hat{V} = \frac{1}{4i\lambda} \begin{pmatrix} \cos\phi & -i\sin\phi \\ i\sin\phi & -\cos\phi \end{pmatrix} \;,$$

- The problem is isospectral: $\lambda = const$ if the initial condition evolves as a SGE $\phi_{xt} = \sin\phi$ solution.
- Consistency condition: $\frac{\partial \hat{U}(\lambda)}{\partial t} - \frac{\partial \hat{V}(\lambda)}{\partial x} - \left[\hat{U}(\lambda), \hat{V}(\lambda)\right] = 0 \leftrightarrow .\phi_{xt} = \sin\phi$
- Inverse problem - reconstruction of the time evolution of $\phi(x,t)$ from the scattering data $\psi(x,t,\lambda)$ and spectrum λ_n using so called Gelfand-Levitan-Marchenko (GLM) equation.

- IST method - transfer to the new “action-angle” equation.
- Topological soliton (+) and antisoliton(-):

$$\phi(x,t) = 4\arctan e^{\pm\frac{x-vt}{\sqrt{1-v^2}}}$$

- Impossible to remove by local deformations.
- Soliton has clearly defined charged particle properties, in particular
  - Charge: $Q = \pm 1$;
  - Momentum: $P = 8v/\sqrt{1-v^2}$;
  - Energy: $E = 8/\sqrt{1-v^2}$.
- Two solitons with same polarity ($Q = 1$) pass through each other without change;
- Two solitons with opposite polarities ($Q = \pm 1$) form a bound state - bion or breather

$$\phi(x,t) = 4\arctan\left\{\frac{\mu}{\omega}\frac{\sin(\omega t - qx - \psi_0)}{\cosh[\mu(x - \bar{x}_0 - vt)]}\right\},\quad \omega^2 + \mu^2 = \frac{1}{1-v^2},\ q = \omega v.$$

If $v = 0$ - immobile localized vibration with $\omega < 1$.

Dissipative effects and external bias

Two types of dissipation exist:

- Normal dissipation - caused by the electron motion across the junction, $i_{R,\perp} = \frac{V}{R_\perp} = \frac{\hbar}{2eR_\perp}\frac{\partial\phi}{\partial t}$;
- Surface dissipation - caused by the electron motion along the junction, $\frac{1}{R_{||}}\frac{\partial^2 V}{\partial x^2} = \frac{\hbar}{2eR_{||}}\frac{\partial^3\phi}{\partial t\partial x^2}$.

$$\frac{\hbar}{2eL}\frac{\partial^2\phi}{\partial x^2} + \frac{1}{R_{||}}\frac{\partial^2 V}{\partial x^2} = C\frac{\partial V}{\partial t} + I_c\sin\phi + \frac{\hbar}{2eR_\perp}\frac{\partial\phi}{\partial t} + I_B\ .$$

In the dimensional units $x \to x/\lambda_J$, $t \to t\omega_J$:

$$\beta\phi_{xxt} + \phi_{xx} - \phi_{tt} - \alpha\phi_t = \sin\phi + \gamma\ .$$

where

$$\alpha = \frac{1}{R_\perp}\sqrt{\frac{\Phi_0}{2\pi I_c C}}\ ,\quad \beta = \sqrt{\frac{2\pi L^2 I_c}{R_{||}^2 C\Phi_0}}\ .$$

Soliton motion in the long junction

Consider the LJJ with the length $\to \infty$ with a soliton inside.
Total energy

$$\frac{dH(\phi)}{dt} = -\int_{-\infty}^{+\infty} (\alpha\phi_t^2 + \beta\phi_{xt}^2)dx, \quad H(\phi) = H^{SG} + H^P = \int_{-\infty}^{+\infty} \left[\frac{\phi_t^2 + \phi_x^2}{2} + 1 - \cos\phi + \gamma\phi\right]$$

decreases because $\alpha, \beta > 0$.
Terms with α, β, γ - perturbation. Assumption: the perturbation is so weak and slow, that does not change the soliton shape. Only the parameter(s) evolve in time (due to perturbation).
Since $H^{SG}(\phi_0) = 8/\sqrt{1 - v^2}$, assuming $v = v(t)$, we obtain

$$\frac{8v\frac{dv}{dt}}{(1 - v^2)^{3/2}} = -\int_{-\infty}^{+\infty} (\gamma\phi_{0t} + \alpha\phi_{0t}^2 + \beta\phi_{0xt}^2)dx \to \frac{dv}{dt} = \pm\frac{\pi\gamma}{4}(1 - v^2)^{3/2} - \alpha v(1 - v^2) - \frac{\beta}{3}v.$$

In the limit $t \to +\infty$ equilibrium velocity (if $\beta = 0$)

$$v(t \to +\infty) \equiv v_\infty = \pm\mathrm{sign}(\gamma)\left[1 + \left(\frac{4\alpha}{\pi\gamma}\right)^2\right]^{-1/2}.$$

Reference: McLaughlin, Scott, Phys. Rev. A 1978.

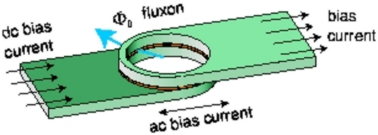
CVC of the long junction

- Linear junction with length l, boundary conditions: $\phi_x(0,t) = \phi_x(l,t) = 0$,
- Soliton shuttle - soliton reflects from the junction edges and changes polarity each time.
- Average voltage drop (dimensionless):

$$\langle V\rangle_T = \frac{1}{l}\lim_{t\to+\infty}\frac{1}{t}\int_0^t\int_0^l \frac{\partial\phi(x',t')}{\partial t}dx'dt' = \frac{1}{T}\int_0^T \frac{\partial\phi(x',t')}{\partial t}dt' = \frac{2\pi v_\infty}{l}$$

- Ring junction of length $l = 2\pi R$, $l \gg w$.
- Periodic b.c. ("+" for soliton and "-" for antisoliton): $\phi(x,t) \pm 2\pi = \phi(x+l,t)$.
- At $t \to +\infty$ the soliton will circumvent the junction during the time $T = l/v_\infty$. Thus,

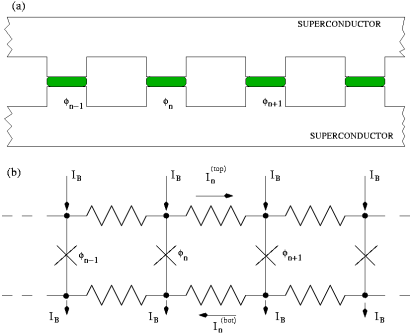
$$\langle V\rangle_T = \frac{2\pi v_\infty}{l} = \frac{v_\infty}{R}.$$



First experiment - Davidson *et al*, Phys. Rev. Lett. 1985.

Josephson junction array

Consider a parallel array of identical JJs (see Watanabe et al, PRL 1996)



- Josephson equations + Kirchhoff laws:

$$\frac{C\hbar}{2e}\ddot{\phi}_n + \frac{\hbar}{2eR}\dot{\phi}_n + I_c \sin\phi_n =$$
$$= I_B - I_n^{top} + I_{n-1}^{top} = I_B - I_n^{bot} + I_{n-1}^{bot}$$

Flux quantisation $\phi_{n+1} - \phi_n = 2\pi\Phi_n/\Phi_0$ and $\Phi_n = -(L_1 I_n^{top} + L_2 I_n^{bot})$, where $L_{1,2}$ - are the inductances of the top/bottom parts of the cell.
Discrete sine-Gordon equation (DSG)

$$\ddot{\phi}_n - \kappa\Delta\phi_n + \sin\phi_n + \alpha\dot{\phi}_n = \gamma = \frac{I_B}{I_c},\ n \in \mathbb{Z}\ ,\quad \Delta\phi_n \equiv \phi_{n+1} - 2\phi_n + \phi_{n-1}.$$

coupling constant $\kappa = \sqrt{\Phi_0/[2\pi I_c L]}$ measures the degree of system discreteness.

Josephson junction array

- In the continuum limit $[\phi_n(t) \to \phi(x,t)]$ DSG becomes standard SG equation.
- Dispersion law for plane waves becomes periodic:

$$\omega_L(q) = \sqrt{1 + 4\kappa \sin^2 q/2}.$$

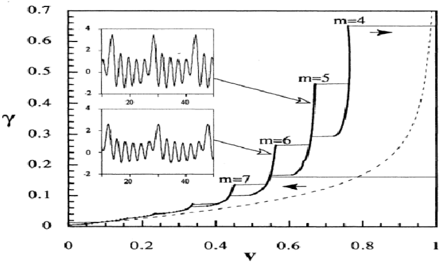
- Discreteness obstructs free soliton motion.
- Fluxon moving with any velocity v will excite a plane wave with the same phase velocity.
- Soliton as a particle will feel the lattice as a spatially periodic potential.
- Dynamics of the discrete soliton can be considered as a particle in the so-called Peierls-Nabarro potential:

$$\ddot{X} + \alpha \dot{X} + V'_{PN}(X) = \gamma \quad , V_{PN}(X).$$

similar to the damped and driven pendulum.

Josephson junction array

- Discreteness: single curve splits into multiple ones.
- The consequence of periodic b.c. and resonances $[\omega_L(q) = vq]$.
- A finite number of phase ϕ oscillations will fit into one cycle of the fluxon journey around the array.



From Ustinov, Cirillo, Malomed, PRB (1993).

Josephson junction array

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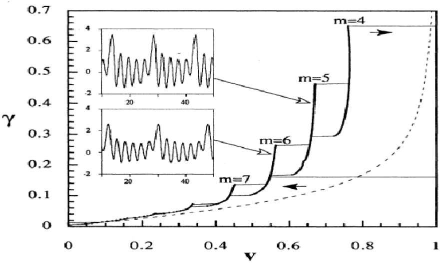

From Ustinov, Cirillo, Malomed, PRB (1993).

Applications

Fluxon-based qubit

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Josephson Vortex Qubit: Design, Preparation and Read-Out

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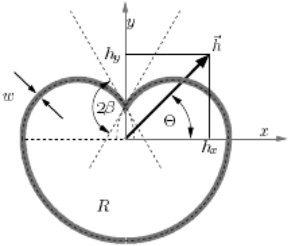

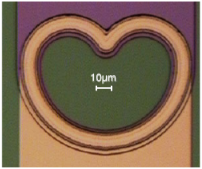

Fig. 8 (online colour). Optical microscope picture of the junction used for the experimental test. Parameters are $R = 50$ μm, $\beta = 60^\circ$, $w = 3$ μm, $j_c = 796$ A/cm^2

9. Experimental Test We carried out an experimental test of the preparation and readout scheme proposed above using the junction shown in Fig. 8.

Figure 9 shows the measured depinning current in dependence on the angle Θ of

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Reading out the state of a flux qubit by Josephson transmission line solitons

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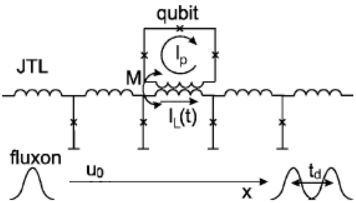

FIG. 1. Setup for the readout of the persistent current qubit based on the delay time of a soliton in the Josephson transmission line (JTL).

Applications

And much more:

- Magnetic field measurement - SQUIDs.
- Voltage standard, Kautz, App. Phys. Lett, 1980.
- A low-noise front-end detector in the range from 100GHz to 1 THz used in radioastronomy (Koshelets et al, IEEE Trans. Appl. Supercond, 1995-97).

Thank you!