Superconductivity and Electronic Structure (Exercises)

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Task 1: GINZBURG-LANDAU THEORY

The basic idea behind Ginzburg-Landau (GL) theory of superconductivity (1950) was to write the free energy as a simple functional of the order parameter of a thermodynamic system and their derivatives. The GL theory is formulated in terms of the complex order parameter $\psi(\mathbf{r})$ (now we understand that it is the wave function of a Cooper pair) which may be written in the form of a product involving a phase factor $\phi(\mathbf{r})$ and a modulus $|\psi(\mathbf{r})|$ where $|\psi(\mathbf{r})|^2 = n_s(\mathbf{r})$ is the super-electron density:

$$\psi(\mathbf{r}) = |\psi(\mathbf{r})| \exp(i\phi(\mathbf{r})). \tag{1}$$

The free-energy density of a superconductor can be expressed in terms of the expansion in this quantity:

$$f_s - f_n = \alpha|\psi(\mathbf{r})|^2 + \beta|\psi(\mathbf{r})|^4 + \frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\nabla + \frac{e^*}{c}\mathbf{A}\right)\psi(\mathbf{r})\right|^2, \tag{2}$$

where the subscripts n and s refer to the normal and superconducting states, respectively, $\mathbf{A}$ is the magnetic vector potential, the magnetic field

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{3}$$

Later, it will be clear that $m^* = 2m$ and $e^* = 2e$, where 2 reflects the number of electrons in Cooper pairs.

The free energy (2) is a functional of the order parameter $\psi(\mathbf{r})$, meaning the actual value of the order parameter realized in equilibrium satisfies $\delta f/\delta\psi(\mathbf{r}) = 0$.

(a) Show that for a uniform system in a zero field a minimum of the free energy with a nonzero value becomes possible when α changes sign.

In the GL theory we are interested in the region near the critical temperature of the superconducting-normal transition T_c, Thus we may take only the leading terms in the Taylor series expansions in this region: $\alpha(T) = \alpha_0(T - T_c)$ and $\beta = const.$

(b) Find the differences in the free energies f_s and f_n below and above T_c and show that the transition to the superconducting state is energetically favorable below T_c.

Note that the energy of the superconducting state below T_c is always lower than that of the normal state by an amount called the *condensation energy*.

Now let us ask what will happen if we apply a weak magnetic field described by $\mathbf{A}$ to the system. Since it is a small perturbation, we do not expect it to couple to $|\psi(\mathbf{r})|$ but rather to the phase $\phi(\mathbf{r})$.

(c) Show that the third term in Eq.(2) represents a kinetic energy of the system and equals to $f_{kin} = \frac{1}{2}m^* n_s^* v_s^2$, where $n_s^* = \frac{1}{2}n_s$ and the superfluid velocity $v_s = \frac{1}{m_s^*}\left(\nabla\phi(\mathbf{r}) + \frac{e^*}{c}\mathbf{A}\right)$.

(d) Compare the obtained result with the general formula for the probability current in quantum mechanics $\mathbf{j} = \frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$.

Next, we obtain

$$\nabla \times \mathbf{j}_s = -\frac{e^{*2} n_s^*}{m^* c}\mathbf{B}$$

and recalling the Maxwell equation $\mathbf{j}_s = \frac{c}{4\pi}\nabla \times \mathbf{B}$ we get

$$\lambda_L \nabla^2 \mathbf{B} = \mathbf{B}, \tag{4}$$

where λ_L is known as a London penetration depth.

(e) Find the expression connecting λ_L with m^*, e^* and n_s^*, analyze the temperature behavior of the magnetic penetration depth using the temperature dependence of the parameter α.

If one considers a superconducting half-space, i.e., a superconductor for $x > 0$, and weak external magnetic field $\mathbf{B}$ applied along z direction in the empty space $x < 0$, then inside the superconductor the magnetic field decays exponentially with the characteristic length scale given by λ_L.

(f) Prove the latter statement which is known as a Meissner effect. Analyze the temperature dependence of the magnetic penetration depth using the temperature dependence of the parameter α.

Note that accurate and precise measurements of the absolute value of penetration depth at low temperatures are very important to understand the mechanism of superconductivity.

Task 2: COOPER PAIRS

Let us discuss the ground state of an electron gas where at zero temperature all one-electron orbitals with wave vectors $k < k_F$ (k_F is the Fermi wave vector) are occupied, and all the rest are empty. One year before publication of the BCS theory, Cooper (1956) demonstrated that such normal-state ground state is unstable with respect to the formation of bound electron pairs when a weak attractive interaction exists between the electrons. The effect of the interaction will be to scatter electrons from (k_1, k_2) states to states with wave vectors (k_1', k_2'). Clearly, the scattering processes tend to increase the kinetic energy of the system. However, as was shown by Cooper, the increase in kinetic energy is more than compensated by a decrease in the potential energy if we allow states above k_F to be occupied in the many-electron ground state. Let us try to prove it within this task.

Consider a pair of electrons in a singlet state which is described by the following wave function

$$\psi(\mathbf{r}_1 - \mathbf{r}_2) = \int \frac{d\mathbf{k}}{(2\pi)^3} \chi(\mathbf{k}) \exp(i\mathbf{k}(\mathbf{r}_1 - \mathbf{r}_2)). \tag{5}$$

The Schroedinger equation for two electrons interacting via the potential V reads as

$$\left[-\frac{\hbar^2}{2m}(\nabla_1^2 + \nabla_2^2) + V(\mathbf{r}_1 - \mathbf{r}_2) \right] \psi(\mathbf{r}_1 - \mathbf{r}_2) = E\psi(\mathbf{r}_1 - \mathbf{r}_2). \tag{6}$$

Here the energy eigenvalue E is defined relative the Fermi level $2E_F$.

(a) Show that in the momentum representation the Schroedinger equation takes the form

$$\left(E - 2\frac{\hbar^2 k^2}{2m} \right) \chi(\mathbf{k}) = \int \frac{d\mathbf{k}}{(2\pi)^3} V(\mathbf{k}, \mathbf{k}') \chi(\mathbf{k}'). \tag{7}$$

The presence of a degenerate free-electron gas is felt only via the exclusion principle. As was stated above, electron levels with $k < k_F$ are forbidden to each of the two electrons, which gives the constraint:

$$\chi(\mathbf{k}) = 0, k < k_F. \tag{8}$$

We suppose a simplest attractive form for the attractive potential of the pair

$$\begin{aligned} V(\mathbf{k}, \mathbf{k}') &= -V, & E_F \leq \frac{\hbar^2 k_1^2}{2m}, \frac{\hbar^2 k_1^2}{2m} \leq E_F + \hbar\omega \\ V(\mathbf{k}, \mathbf{k}') &= 0, & \text{otherwise,} \end{aligned}$$

where ω is the upper frequency of the phonon spectrum and look for a bound state with the energy E less than $2E_F$. The binding energy will be

$$\Delta = 2E_F - E. \tag{9}$$

(b) Show that a bound state of energy E exists provided by the expression

$$V \int_{E_F}^{E_F+\hbar\omega} \frac{N(E')dE'}{2E' - E} = 1, \tag{10}$$

where $N(E)$ is the density of one-electron levels of a given spin.

(c) Show that Eq.10 has the solution with $E < 2E_F$ for arbitrarily weak V in the case if $N(E_F)$ is not zero.

(d) Assuming that $N(E) = N(E_F)$ for $E_F \leq E \leq E_F + \hbar\omega$ show that the binding energy is given by

$$\Delta = 2\hbar\omega \frac{\exp(-2/N(E_F)V)}{1 - \exp(-2/N(E_F)V} \approx 2\hbar\omega \exp(-2/N(E_F)V)$$

for a weak attractive potential.